# Subsurface Electromagnetic Fields of a Grounded Cable of Finite Length[1]

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The subsurface fields of a finite line source or current-carrying cable are examined. Some special cases, such as the low-frequency limit, are treated analytically, and some simple working formulas are obtained. The general field expressions are reduced to single integrals with finite limits that are evaluated numerically. It is shown that if the source cable is sufficiently long, the fields are approximated by those for an infinite cable. The results have possible application to downlink communication and radio location of trapped miners.

On étudie les champs produits sous la surface du sol par une source constituée par une portion rectiligne finie d'un câble transportant du courant. Certains cas spéciaux, par exemple, la limite aux basses fréquences, sont traités analytiquement, et quelques formules directement applicables sont obtenues. Les expressions générales des composantes de champ sont réduites à des intégrales simples avec des limites finies, et ces intégrales sont évaluées numériquement. On montre que si le câble source est suffisamment long, les champs sont approximativement les mêmes que ceux d'un câble infini. Les résultats ont des applications pour les communications souterraines et la localisation par radio de mineurs emprisonnés.

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## Introduction

The subsurface fields of an infinite line source on a conducting half-space have been examined by Wait and Spies (1971). In any real communication link from the surface to an underground terminal, the line source or current-carrying cable is of finite length. Here, we examine the subsurface fields of a finite line source with the purpose of determining what cable length is sufficient, and when the infinite line source is a good approximation. The earth is taken to be a homogeneous conducting half-space, and the current in the cable is assumed to be constant. The latter assumption is well justified at audio frequencies when an insulated cable is terminated by grounded electrodes that are separated by distances small compared with a free-space wavelength (Wait 1952; Sunde 1968).

## Formulation

The geometry of the line source of length $2l$ located on the half-space, and the observer in the half-space, are as shown in Fig. 1. The observer is located at $(x, y, z)$, and the $x$-directed source current $I$ runs from $-l$ to $l$ on the $x$ axis. The current varies as $\exp(i\omega t)$, and the angular frequency $\omega$ is sufficiently low that all displacement currents are negligible. The Hertz vector contains both $x$ and $z$ components ($\Pi_x$ and $\Pi_z$) and the contribution from an incremental source of length $dx'$ located at $x'$ is given by Wait (1961) and Banos (1966):

$$\mathrm{d}\Pi_x = \frac{I\,\mathrm{d}x'}{2\pi\sigma}\int_0^{\infty}\frac{e^{uz}}{u+\lambda}J_0(\lambda\rho)\lambda\,\mathrm{d}\lambda \tag{1}$$

$$\mathrm{d}\Pi_z = \frac{I\,\mathrm{d}x}{2\pi\sigma}\frac{\partial}{\partial x}\int_0^{\infty}\frac{e^{uz}}{u+\lambda}J_0(\lambda\rho)\,\mathrm{d}\lambda$$

where

$$u = (\lambda^2+\gamma^2)^{1/2}, \quad \gamma^2 = i\omega\mu_0\sigma, \quad \rho = [y^2+(x-x')^2]^{1/2}$$

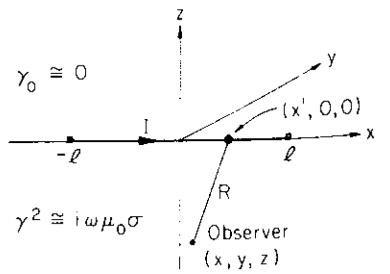


FIG. 1. Finite line source on a homogeneous half-space.

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$\mu_0$ is the permeability of free space, and $\sigma$ is the conductivity of the half-space. Here, and in what follows, we restrict the observer to the subsurface region $z < 0$.

Since a numerical integration on $x'$ will be required, it is desirable to eliminate the $\lambda$ integration. For this quasi-static situation, Wait (1961) has shown that the $\lambda$ integration can be performed to yield the result

$$[2] \quad \begin{aligned} d\Pi_x &= \frac{I\,dx'}{2\pi\sigma\gamma^2}\left(\frac{\partial^2 P}{\partial z^2} - \frac{\partial^3 N}{\partial z^3} + \gamma^2\frac{\partial N}{\partial z}\right) \\ d\Pi_z &= \frac{I\,dx'}{2\pi\sigma\gamma^2}\left(\frac{\partial^3 N}{\partial x\partial z^2} - \frac{\partial^2 P}{\partial x\partial z}\right) \end{aligned}$$

where $N = I_0[(\gamma/2)(R + z)]K_0[(\gamma/2)(R - z)]$, $P = R^{-1}\exp(-\gamma R)$, $R = (\rho^2 + z^2)^{1/2}$, and $I_0$ and $K_0$ are modified Bessel functions of order zero. The magnetic field components are derived from the curl of the Hertz vector

$$[3] \quad d\boldsymbol{H} = \sigma \operatorname{curl} d\boldsymbol{\Pi}$$

By carrying out the curl operation in [3], and utilizing the fact that $N$ satisfies the wave equation, $(\nabla^2 - \gamma^2)N = 0$, the magnetic field components are obtained:

$$[4] \quad \begin{aligned} dH_x &= \frac{I\,dx'}{2\pi\gamma^2}\left(\frac{\partial^4 N}{\partial x\partial y\partial z^2} - \frac{\partial^3 P}{\partial x\partial y\partial z}\right) \\ dH_y &= \frac{I\,dx'}{2\pi\gamma^2}\left(\frac{\partial^3 P}{\partial z^3} + \frac{\partial^3 P}{\partial x^2\partial z} + \frac{\partial^4 N}{\partial z^2\partial y^2}\right) \\ dH_z &= \frac{I\,dx'}{2\pi\gamma^2}\left(\frac{\partial^4 N}{\partial y\partial z^3} - \gamma^2\frac{\partial^2 N}{\partial y\partial z} - \frac{\partial^3 P}{\partial y\partial z^2}\right) \end{aligned}$$

The electric field components have been given by Wait (1961):

$$[5] \quad \begin{aligned} dE_x &= \frac{-I\,dx'}{2\pi\sigma}\left(\frac{\partial^2 P}{\partial z^2} + \frac{\partial^3 N}{\partial y^2\partial z}\right) \\ dE_y &= \frac{I\,dx'}{2\pi\sigma}\frac{\partial^3 N}{\partial y\partial x\partial z} \\ dE_z &= \frac{I\,dx'}{2\pi\sigma}\frac{\partial^2 P}{\partial x\partial z} \end{aligned}$$

To obtain the resultant fields for the finite-length cable, we integrate [4] and [5] over the range of $x'$ from $-l$ to $l$.

The components of interest here are $H_y$, $H_z$, and $E_x$, since all other components vanish in the vertical plane $x = 0$. Also, these are the only nonzero components everywhere if the line source is of infinite length. For normalization purposes, it is useful to write the fields in the manner

$$[6] \quad \begin{aligned} H_y &= \frac{I}{2\pi h}A(H, Y, X, L) \\ H_z &= \frac{I}{2\pi h}B(H, Y, X, L) \\ E_x &= \frac{-i\omega\mu_0 I}{2\pi}F(H, Y, X, L) \end{aligned}$$

where $H = (\omega\mu_0\sigma)^{1/2}h$, $Y = y/h$, $X = x/h$, and $L = l/h$ and $h(= -z)$ is the observer depth. Note that $A$, $B$, and $F$ are dimensionless. For the case of the infinite line source ($l = \infty$), these quantities have been tabulated (Wait and Spies 1971). By comparing [4] and [5] with [6], the following integral forms are found for $A$, $B$, and $F$:

$$[7] \quad \begin{aligned} A(H, Y, X, L) &= \frac{h}{\gamma^2}\int_{-l}^{l}\left(\frac{\partial^3 P}{\partial z^3} + \frac{\partial^3 P}{\partial x^2\partial z} + \frac{\partial^4 N}{\partial z^2\partial y^2}\right)dx' \\ B(H, Y, X, L) &= \frac{h}{\gamma^2}\int_{-l}^{l}\left(\frac{\partial^4 N}{\partial y\partial z^3} - \gamma^2\frac{\partial^2 N}{\partial y\partial z} - \frac{\partial^3 P}{\partial y\partial z^2}\right)dx' \\ F(H, Y, X, L) &= \frac{1}{\gamma^2}\int_{-l}^{l}\left(\frac{\partial^2 P}{\partial z^2} + \frac{\partial^3 N}{\partial y^2\partial z}\right)dx' \end{aligned}$$

The specific expressions for the partial derivatives of $P$ and $N$ in [7] are given in the Appendix.

## Low-Frequency Limit

The low-frequency limit is useful both as a check on the numerical work and as a simple means of determining trends at low frequencies. If the frequency approaches zero, then $\gamma$ approaches zero and the Hertz components are given by

$$[8] \quad \begin{aligned} d\Pi_x &= \frac{I\,dx'}{4\pi\sigma}\int_0^\infty e^{\lambda z}J_0(\lambda\rho)\,d\lambda = \frac{I\,dx'}{4\pi\sigma}\frac{1}{R} \\ d\Pi_z &= \frac{-I\,dx'}{4\pi\sigma}\frac{(x - x')}{\rho}\int_0^\infty e^{\lambda z}J_1(\lambda\rho)\,d\lambda \\ &= \frac{-I\,dx'}{4\pi\sigma}\frac{(x - x')}{\rho^2}(1 + z/R) \end{aligned}$$

where we make use of the integrals listed by Wheelon (1968). It is important to note that $d\Pi_z$ does not vanish in this limit. This is consistent with the notion that the electric fields can be derived from a scalar potential $\Phi$ that is given by

$$\Phi = -\left(\frac{\partial \Pi_x}{\partial x} + \frac{\partial \Pi_z}{\partial z}\right) = \frac{-I\,dx'}{2\pi\sigma}\frac{\partial}{\partial x}\left(\frac{1}{R}\right)$$

for the static limit.

The magnetic field components are obtained from the curl operation in [3]. Thus

$$[9]\quad dH_z = \frac{I\,dx'}{4\pi}\frac{y}{R^3}, \qquad dH_y = \frac{I\,dx'}{4\pi}\left\{\frac{-z}{R^3} + \frac{\partial}{\partial x}\left[\frac{x-x'}{\rho^2}\left(1+\frac{z}{R}\right)\right]\right\}$$

There is also an $x$ component which is not of interest here. The quantities in [9] can be integrated from $x' = -l$ to $x' = +l$ to obtain the total fields

$$[10]\quad \begin{aligned} H_y = \frac{I}{4\pi}\Bigg\{ & \frac{-z}{y^2+z^2}\left[\frac{l-x}{[(l-x)^2+y^2+z^2]^{1/2}} + \frac{l+x}{[(l+x)^2+y^2+z^2]^{1/2}}\right] \\ & + \frac{l-x}{(l-x)^2+y^2}\left[1+\frac{z}{[(l-x)^2+y^2+z^2]^{1/2}}\right] \\ & + \frac{l+x}{(l+x)^2+y^2}\left[1+\frac{z}{[(l+x)^2+y^2+z^2]^{1/2}}\right]\Bigg\} \\ H_z = \frac{I}{4\pi} & \frac{y}{y^2+z^2}\left[\frac{l-x}{[(l-x)^2+y^2+z^2]^{1/2}} + \frac{l+x}{[(l+x)^2+y^2+z^2]^{1/2}}\right] \end{aligned}$$

From [10], the normalized quantities $A$ and $B$, as defined in [6], are found to be

$$[11]\quad \begin{aligned} A(0, Y, X, L) = & \frac{-1}{2(1+Y^2)}\left[\frac{L-X}{[(L-X)^2+1+Y^2]^{1/2}} + \frac{L+X}{[(L+X)^2+1+Y^2]^{1/2}}\right] \\ & + \frac{(L-X)/2}{(L-X)^2+Y^2}\left[1-\frac{1}{[(L-X)^2+1+Y^2]^{1/2}}\right] \\ & + \frac{(L+X)/2}{(L+X)^2+Y^2}\left[1-\frac{1}{[(L+X)^2+1+Y^2]^{1/2}}\right] \\ B(0, Y, X, L) = & \frac{Y/2}{1+Y^2}\left[\frac{L-X}{[(L-X)^2+1+Y^2]^{1/2}} + \frac{L+X}{[(L+X)^2+1+Y^2]^{1/2}}\right] \end{aligned}$$

As $L$ approaches $\infty$, the following results are obtained:

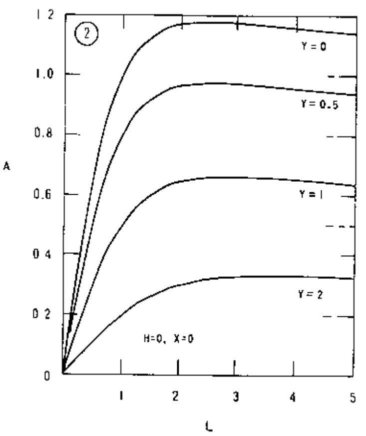
$$[12]\quad A(0, Y, X, \infty) = \frac{1}{1+Y^2}, \quad B(0, Y, X, \infty) = \frac{Y}{1+Y^2}$$

The results are of course independent of $X$ and agree with infinite line source results (Wait and Spies 1971).

Results for finite $L$ are shown in Figs. 2 and 3 for $X = 0$. Note that $A$ in Fig. 2 overshoots its final value and that $L = 1$ is sufficient to obtain a horizontal magnetic field equal to that of the infinite line source. However, the $B$ values in Fig. 3 approach their final values quite slowly, particularly for larger $Y$. Consequently, a fairly large value of $L$ is required to obtain a vertical magnetic field nearly equal to that of the infinite line source.

## High-Frequency Limit

The other useful limiting case is for high frequencies where $|\gamma y|$ is large. In this case, terms involving $P$ are negligible, and $N$ is asymptotically given by (Wait 1961)



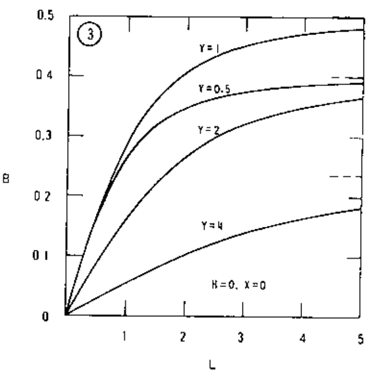




FIG. 2. Low-f[illegible]ncy limit of the horizontal magnetic field as a function of $Y$ and $L$. Final values are dashed.
FIG. 3. Low-frequency limit of the vertical magnetic field as a function of $Y$ and $L$. Final values are dashed.

$$N \sim \frac{e^{\gamma z}}{\gamma \rho} \quad \text{for} \quad z \leqslant 0 \tag{13}$$

Consequently, $dE_x$ and $dH_y$ are given by

$$dE_x \sim -\frac{I\,dx'}{2\pi\sigma} e^{\gamma z} \frac{\partial^2}{\partial y^2}\left(\frac{1}{\rho}\right) \quad \text{and} \quad dH_y \sim -\frac{I\,dx'}{2\pi\gamma} e^{\gamma z} \frac{\partial^2}{\partial y^2}\left(\frac{1}{\rho}\right) \tag{14}$$

In order to determine $dH_z$, we must include higher-order terms in $N$. If this is done, the result is

$$dH_z \sim -\frac{I\,dx'}{2\pi\gamma^2} e^{\gamma z} \frac{\partial}{\partial y}\left(\frac{1}{\rho^3}\right) \tag{15}$$

The $x'$ integrations can be done in [14] and [15] to yield the total fields:

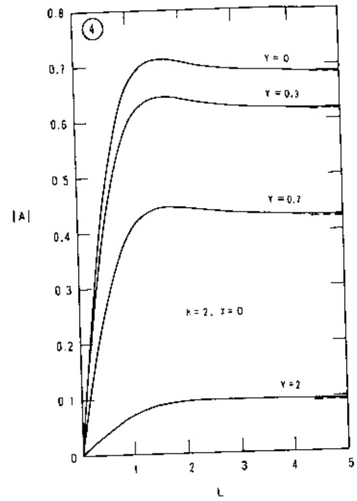
$$\begin{aligned} E_x &\sim -\frac{I}{2\pi\sigma} e^{\gamma z} \frac{\partial^2}{\partial y^2} \{\ln [l - x + [(l-x)^2 + y^2]^{1/2}] \\ &\qquad - \ln [-(l+x) + [(l+x)^2 + y^2]^{1/2}]\} \\ H_y &\sim -\frac{I}{2\pi y} e^{\gamma z} \frac{\partial^2}{\partial y^2} \{\ln [l - x + [(l-x)^2 + y^2]^{1/2}] \\ &\qquad - \ln [-(l+x) + [(l+x)^2 + y^2]^{1/2}]\} \\ H_z &\sim -\frac{I}{2\pi\gamma^2} e^{\gamma z} \frac{\partial}{\partial y} \left\{\frac{1}{y^2}\left[\frac{l-x}{[(l-x)^2 + y^2]^{1/2}} + \frac{l+x}{[(l+x)^2 + y^2]^{1/2}}\right]\right\} \end{aligned} \tag{16}$$

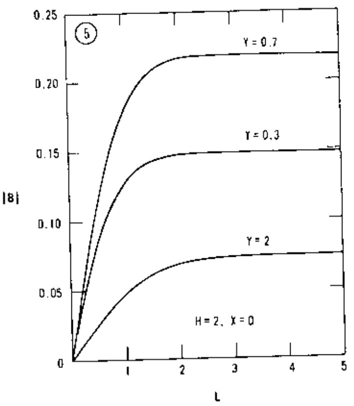
If $l$ approaches $\infty$, then [16] reduces to

$$E_x|_{l=\infty} \sim \frac{-I}{\pi\sigma}\frac{e^{\gamma z}}{y^2}, \quad H_y|_{l=\infty} \sim \frac{I}{\pi\gamma}\frac{e^{\gamma z}}{y^2}, \quad H_z|_{l=\infty} \sim \frac{2I}{\pi\gamma^2}\frac{e^{\gamma z}}{y^3} \tag{17}$$

Thus, the normalized quantities $A$, $B$, and $F$ are given by

$$\begin{aligned} A(H, Y, X, \infty) &\sim \frac{2}{i^{1/2} H Y^2} \exp(-i^{1/2} H) \\ B(H, Y, X, \infty) &\sim \frac{4}{i H^2 Y^3} \exp(-i^{1/2} H) \\ F(H, Y, X, \infty) &\sim \frac{2}{i H^2 Y^2} \exp(-i^{1/2} H) \end{aligned} \tag{18}$$



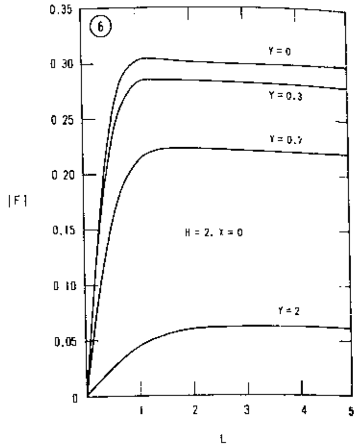




FIG. 4. Magnitude of the horizontal magnetic field as a function of $Y$ and $L$.
FIG. 5. Magnitude of the vertical magnetic field as a function of $Y$ and $L$.
FIG. 6. Magnitude of the horizontal electric field as a function of $Y$ and $L$.

These results agree with those of the infinite line source (Wait and Spies 1971).

We stress that the "high-frequency" limit solutions discussed in this section are applicable when the horizontal distance, ρ, is much *greater* than a skin depth in the earth, and much *less* then the free-space wavelength. Thus, the formulas are of limited applicability, although they are useful for checking the results of the numerical integration.

## General Numerical Results

For general values of [illegible] numerical integration over finite limits of the expressions in [7] is required. Even though the integrands are quite complicated, they are well behaved numerically. Consequently, the computer program for $A$, $B$, and $F$ is [illegible] fast, and the influence of the various parameters can be studied easily. Here, we illustrate some of the important features in graphical form.

Results for $H = 2$ and $X = 0$ are shown in Figs. 4, 5, and 6. For large $L$, all quantities approach those of the infinite line source (Wait and Spies 1971) which is reassuring. Although $A$, $B$, and $F$ are now complex, only the magnitudes are plotted since the phases are relatively constant with $L$.

We note that $|A|$ in Fig. 4 still has a slight overshoot before settling down to its final value for large $L$. A similar trend for $|F|$ is shown in Fig. 6. In both cases, $L$ greater than about 1 is sufficient to yield field strengths nearly equal to that of the infinite line source. However, a larger value of $L$ would be required to reach a nearly maximum value of $|B|$ as shown in Fig. 5. Consequently, if the vertical magnetic field is important, a value for $L$ in the neighborhood of 2 might be more appropriate, if we wish to simulate an infinitely long cable. This could be a crucial factor in designing a subsurface location scheme that makes use of the well-defined structure of the magnetic fields produced by an infinite line source.

## Concluding Remarks

Analytical expressions for the subsurface fields of a finite line source have been derived for high-frequency and low-frequency limits. For the general case, a single numerical integration along the source is required. Although a constant current distribution has been assumed, an arbitrary distribution would present no problem since the source integration is done numerically. Also, multiple grounding of the source cable and arrays of such cables could be treated by simple extensions of the present formulation.

Results for low and medium frequencies indicate that the finite line source fields approach those of an infinite line source quite quickly. For instance, $L \simeq 1$ appears to be sufficient for the horizontal electric and magnetic fields, and $L \simeq 2$ is sufficient for the vertical magnetic field.

It is important to realize that the finite line source excites currents in the earth by induction and by conduction. The effect of the latter vanishes in the limit of an infinitely long cable when the magnetic field is measured at the subsurface point. The results shown in Figs. 2–6 for finite $L$ implicitly account for the effect of the conduction currents injected at the electrodes.

In a sequel to this work, we plan to consider the fields produced by a finite source cable located within a coal mine.

## Appendix

There are four terms in derivatives of $P$ which are required in [7]. The differentiation is straightforward, and the resultant forms are

$$\frac{\partial^2 P}{\partial z^2} = \frac{e^{-\gamma R}}{R^2}\left[\frac{1}{R}\left(-1+\frac{3z^2}{R^2}\right)+\gamma\left(-1+\frac{3z^2}{R^2}\right)+\frac{\gamma^2 z^2}{R}\right]$$

$$\frac{\partial^3 P}{\partial z^3} = \frac{z\,e^{-\gamma R}}{R^3}\left[\frac{3}{R^2}\left(3-\frac{5z^2}{R^2}\right)+\frac{3\gamma}{R}\left(3-\frac{5z^2}{R^2}\right)+3\gamma^2\left(1-\frac{2z^2}{R^2}\right)-\frac{\gamma^3 z^2}{R}\right]$$

[19] $$\frac{\partial^3 P}{\partial y\partial z^2} = \frac{y\,e^{-\gamma R}}{R^3}\left[\frac{3}{R^2}\left(1-\frac{5z^2}{R^2}\right)+\frac{3\gamma}{R}\left(1-\frac{5z^2}{R^2}\right)+\gamma^2\left(1-\frac{6z^2}{R^2}\right)-\frac{\gamma^3 z^2}{R}\right]$$

$$\frac{\partial^3 P}{\partial x^2\partial z} = \frac{z\,e^{-\gamma R}}{R^3}\left\{\frac{3}{R^2}\left[1-\frac{5(x-x')^2}{R^2}\right]+\frac{3\gamma}{R}\left[1-\frac{5(x-x')^2}{R^2}\right]+\gamma^2\left[1-\frac{6(x-x')^2}{R^2}\right]-\frac{\gamma^3(x-x')^2}{R}\right\}$$

The four terms in derivatives of $N$ can be simplified by replacing the derivatives of the modified Bessel functions by Bessel functions of zero and first order. This procedure, used previously by Wait and Campbell (1953*a*, *b*), yields the following expressions which are quite suitable for numerical evaluation.

[20] $$\frac{\partial^2 N}{\partial y\partial z} = \frac{\gamma y}{2}[A_{00}I_0K_0 + A_{11}I_1K_1 + A_{01}I_0K_1 + A_{10}I_1K_0]$$

where

$$A_{00} = \frac{\gamma z}{R^2},\quad A_{11} = -\frac{\gamma z}{R^2},\quad A_{01} = \frac{1}{R^2}\left(-1+\frac{z}{R}\right),\quad A_{10} = -\frac{1}{R^2}\left(1+\frac{z}{R}\right)$$

$$\frac{\partial^3 N}{\partial y^2\partial z} = \frac{\gamma}{2}[B_{00}I_0K_0 + B_{11}I_1K_1 + B_{01}I_0K_1 + B_{10}I_1K_0]$$

where

$$B_{00} = \frac{\gamma z}{R}\left(1-\frac{3y^2}{R^2}\right),\quad B_{11} = \frac{\gamma z}{R^2}\left(-1+\frac{3y^2}{R^2}+\frac{2y^2}{\rho^2}\right)$$

$$B_{01} = \frac{1}{R^2}\left[-1+\frac{z(1-\gamma^2y^2)}{R}+\frac{3y^2}{R^2}-\frac{3y^2z}{R^3}\right],\quad B_{10} = \frac{1}{R^2}\left[-1-\frac{z(1-\gamma^2y^2)}{R}+\frac{3y^2}{R^2}+\frac{3y^2z}{R^3}\right]$$

$$\frac{\partial^4 N}{\partial y^2\partial z^2} = \frac{\gamma}{2}[C_{00}I_0K_0 + C_{11}I_1K_1 + C_{01}I_0K_1 + C_{10}I_1K_0]$$

where

$$C_{00} = \frac{\gamma z^2}{R^4}\left[-3+\gamma^2y^2+\frac{15y^2}{R^2}\right]$$

$$C_{11} = \frac{\gamma}{R^2}\left[-2+\frac{z^2(3-\gamma^2y^2)}{R^2}+\frac{6y^2}{R^2}-\frac{15y^2z^2}{R^4}+\frac{2y^2(R^2-2z^2)}{\rho^2R^2}\right]$$

$$C_{01} = \frac{1}{R^3}\left[\frac{z(3 - \gamma^2\gamma^2)}{R} - \gamma^2(y^2 + z^2) - \frac{3z^2(1 - 2\gamma^2 y^2)}{R^2} - \frac{15zy^2}{R^3} + \frac{15y^2z^2}{R^4} + \frac{\gamma^2y^2z(R + z)}{\rho^2}\right]$$

$$C_{10} = \frac{1}{R^3}\left[\frac{z(3 - \gamma^2y^2)}{R} + \gamma^2(y^2 + z^2) + \frac{3z^2(1 - 2\gamma^2 y^2)}{R^2} - \frac{15zy^2}{R^3} - \frac{15y^2z^2}{R^4} + \frac{\gamma^2y^2z(R - z)}{\rho^2}\right]$$

$$\frac{\partial^4 N}{\partial y \partial z^3} = \frac{\gamma y}{2}[D_{00}I_0K_0 + D_{11}I_1K_1 + D_{01}I_0K_1 + D_{10}I_1K_0]$$

where

$$D_{00} = \frac{\gamma z}{R^4}\left(-3 + \frac{15z^2}{R^2} + \gamma^2z^2\right), \quad D_{11} = \frac{\gamma z}{R^4}\left(13 - \frac{15z^2}{R^2} - \gamma^2z^2\right)$$

$$D_{01} = \frac{1}{R^2}\left(\frac{3}{R^2} - \frac{3z}{R^3} - \frac{15z^2}{R^4} + \frac{15z^3}{R^5} - \gamma^2 - \frac{3\gamma^2z}{R} - \frac{\gamma^2z^2}{R^2} + \frac{6\gamma^2z^3}{R^3}\right)$$

$$D_{10} = \frac{1}{R^2}\left(\frac{3}{R^2} + \frac{3z}{R^3} - \frac{15z^2}{R^4} - \frac{15z^3}{R^5} - \gamma^2 + \frac{3\gamma^2z}{R} - \frac{\gamma^2z^2}{R^2} - \frac{6\gamma^2z^3}{R^3}\right)$$

The argument of $I_0$ and $I_1$ is $(\gamma/2)(R + z)$, and the argument of $K_0$ and $K_1$ is $(\gamma/2)(R - z)$.

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